Parametrizing the Neutrino Mixing Matrix

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We propose parametrizations of the neutrino mixing matrix. We discuss several Ansatzes: in some of them $V_{e3} = 0$ but in others V_{e3} typically "comes out" to be of order $\simeq 0.05$.

I. Neutrino Mixing Matrix

Wolfenstein's parametrization [1] of the quark mixing matrix, while not based on any deep theoretical foundation, has proved to be enormously useful in making sense of the data. Something similar is needed to make sense of the neutrino mixing matrix. In reporting the value of a typical mixing angle θ, experimentalists quote variously $\sin\theta$, $\sin 2\theta$, $\tan\theta$, or their squares. It would be helpful to have a more systematic way of quoting experimental values, analogous to what the Wolfenstein parametrization has provided for the quark sector. In the literature, two parametrizations [2,3] of neutrino mixing had been proposed that I know of. Here we would to propose a quite different parametrization. We also suggest a more systematic way for experimentalists to quote their data. As in the Wolfenstein parametrization, our discussion is not based on any deep theoretical foundation, but rather on a phenomenological analysis of the available data. This paper is thus a somewhat old fashioned mix of Ansatz making and data fitting.

Thanks to heroic experimental efforts, the neutrino mixing matrix is now known to take on the value

$$|V| = \begin{pmatrix} 0.72 - 0.88 & 0.46 - 0.68 & < 0.22 \\ 0.25 - 0.65 & 0.27 - 0.73 & 0.55 - 0.84 \\ 0.10 - 0.57 & 0.41 - 0.80 & 0.52 - 0.83 \end{pmatrix} \tag{1}$$

I am citing the report [4] of Gonzalez-Garcia, based in part on the analysis given by Bachall, Gonzalez-Garcia, and Pena-Garay [5]. Here the notation $|V|$ is such that the ij matrix element of the matrix $|V|$ is equal to $|V_{ij}|$.

The mixing matrix V relates the neutrino current eigenstates (denoted by ν_α ($\alpha = e$, μ, τ) and coupled by the W bosons to the corresponding charged leptons) to the neutrino mass eigenstates (denoted by ν_i ($i = 1, 2, 3$)) according to

$$\begin{pmatrix} \nu_e \\ \nu_\mu \\ \nu_\tau \end{pmatrix} = V \begin{pmatrix} \nu_1 \\ \nu_2 \\ \nu_3 \end{pmatrix} \tag{2}$$

We will take the neutrinos to be Majorana [6] as seems likely, so that we have in the Lagrangian the mass term

$$\mathcal{L} = -\nu_\alpha M_{\alpha\beta} C \nu_\beta + h.c. \tag{3}$$

where C denotes the charge conjugation matrix. Thus, the neutrino mass matrix M is symmetric.

For the sake of simplicity we will assume CP conservation so that M is real. With this simplification, M is diagonalized by an orthogonal transformation

$$V^T M V = \begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix} \tag{4}$$

It will become clear to the reader that everything in this paper is easily generalized to the case in which CP is violated so that V is unitary rather than orthogonal, as briefly indicated in section V. Also, we are free to multiply V on the right by some diagonal matrix whose diagonal entries are equal to ± 1. This merely multiplies each of the columns in V by an arbitrary sign. Various possible phases have been discussed in detail in the literature [7,8].

II. An Ansatz for the Neutrino Mixing Matrix

We could suppose either that the entries in V represent a bunch of meaningless numbers (for example, possibly varying from domain to domain in the universe) or that they point to some deeper structure or symmetry. In the latter spirit, let us make a guess of what V might be.

Since V_{e3} appears to be small, let us boldly set it to 0. Next, since $1/\sqrt{2} \sim 0.707$ we will guess that $V_{\mu3} = (0.55 - 0.84) = 1/\sqrt{2}$. Finally, since $1/\sqrt{3} \sim 0.577$ we will set $V_{e2} = (0.46 - 0.68) = 1/\sqrt{3}$. In other words, we propose that we know the upper triangular entries of the matrix V:

$$V = \begin{pmatrix} X & \frac{1}{\sqrt{3}} & 0 \\ X & X & \frac{1}{\sqrt{2}} \\ X & X & X \end{pmatrix}, \tag{5}$$

where X denotes an unknown quantity.

Remarkably, this essentially fixes the mixing matrix V. Once we take the last column to be proportional to $(0, 1, -1)$, orthogonality and our "knowledge" that V_{e2} is $1/\sqrt{3}$ immediately fix the second column to be proportional $(1, 1, 1)$ and hence the first column to be proportional to $(-2, 1, 1)$. X. G. He and I therefore arrived at the Ansatz or guess [9]

$$V = \begin{pmatrix} -\frac{2}{\sqrt{6}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix}. \tag{6}$$

This Ansatz was proposed earlier by Harrison, Perkins and Scott [10]. Also, this mixing matrix (but curiously, with the first and second column interchanged) was first suggested by Wolfenstein more than 20 years ago [11] based on some considerations involving the permutation group S_3. It has subsequently been studied extensively by Harrison, Perkins and Scott [12], and by Xing [13]. Attempts to derive this mixing matrix have been discussed by Low and Volkas [14]. We will refer to the matrix in (6) as V unless confusion could arise in which case we will refer to it as $V_{HPSHZ.}$.

The signs in V were chosen to minimize the number of minus signs. The vanishing of V_{e3} indicates, if this Ansatz is correct, that there is no observable CP violating effects in neutrino oscillation experiments. See (26) below.

The mixing matrix V may be factorized as

$$V = V_{23}V_{12} \tag{7}$$

where

$$V_{23} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{8}$$

and

$$V_{12} = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{1}{\sqrt{3}} & \sqrt{\frac{2}{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{9}$$

Note that our sign choices in (6) is such that $\det V_{23} = \det V_{12} = -1$ (but $\det V = 1$.)

The rotation of the neutrino current eigenstates appear to "occur in two steps." In other words, if we follow Wolfenstein and define $\nu_x \equiv (\nu_\mu + \nu_\tau)/\sqrt{2}$ and $\nu_y \equiv (\nu_\mu - \nu_\tau)/\sqrt{2}$ we find that the mass eigenstates are given by

$$\nu_1 = -\sqrt{\frac{2}{3}}\nu_e + \frac{1}{\sqrt{3}}\nu_x, \tag{10}$$

$$\nu_2 = \frac{1}{\sqrt{3}}\nu_e + \sqrt{\frac{2}{3}}\nu_x, \tag{11}$$

and

$$\nu_3 = \nu_y \tag{12}$$

The matrix V_{12} describes a rotation through $\arcsin(\frac{1}{\sqrt{3}}) \sim 35^o$.

Giunti [15] has proposed another Ansatz, to which we will turn later.

III. Parametrizing the Neutrino Mixing Matrix

In order to compare theory with experiment, we have to understand what the experimental values in (1) mean. The quoted range of numbers are extracted from different experiments, but the error ranges are obviously not uncorrelated since V has to be unitary, or orthogonal with our simplifying assumptions.

There is not a unique way to extract an orthogonal matrix $V_{\exp}$ from (1). Of the many possible ways we choose the following procedure. We take the mean value of the cited range of the various entries in $|V|$ (e.g. for $|V_{e2}|$ we set $(0.46 - 0.68) \to 0.57$), fix the signs of various entries to agree with the choices in (6), and thus arrive at the "experimental mean values"

$$V_{\mathrm{mean}} = \begin{pmatrix} -0.8 & 0.57 & 0.00 \\ 0.45 & 0.5 & 0.695 \\ 0.335 & 0.605 & -0.675 \end{pmatrix} \tag{13}$$

The matrix V_{mean} is not orthogonal. Since $V_{\mathrm{mean}}^T V_{\mathrm{mean}}$ is real symmetric it can be diagonalized by an orthogonal matrix S so that the matrix $K^2 \equiv S^T V_{\mathrm{mean}}^T V_{\mathrm{mean}} S$ is diagonal with positive definite diagonal elements. We then define the "corrected experimental mixing matrix" to be the orthogonal matrix

$$V_{\exp}^{\mathrm{corr}} \equiv V_{\mathrm{mean}}(SK^{-1}S^T). \tag{14}$$

Numerically, this comes out to be

$$V_{\exp}^{\mathrm{corr}} = \begin{pmatrix} -0.814 & 0.578 & 0.056 \\ 0.436 & 0.545 & 0.716 \\ 0.384 & 0.607 & -0.696 \end{pmatrix} \tag{15}$$

Note that a non-zero value of V_{e3} is "generated" but one order of magnitude smaller than the other entries. Second, note that $V_{\exp}^{\mathrm{corr}}$ is well within the range quoted in (1). Finally, compare $V_{\exp}^{\mathrm{corr}}$ with the theoretical Ansatz (6)

$$V = \begin{pmatrix} -0.817 & 0.577 & 0.000 \\ 0.408 & 0.577 & 0.707 \\ 0.408 & 0.577 & -0.707 \end{pmatrix} \tag{16}$$

Clearly, V and $V_{\exp}^{\mathrm{corr}}$ are rather close. Thus, three obvious ways of parametrizing the data suggest themselves. We could parametrize the data by a matrix W defined by $V_{\exp}^{\mathrm{corr}} = VW$, or a matrix X defined by $V_{\exp}^{\mathrm{corr}} = XV$, or a matrix Y defined by $V_{\exp}^{\mathrm{corr}} = V_{23}YV_{12}$ (as suggested by (7)). Namely, we parametrize the deviation of V from $V_{\exp}^{\mathrm{corr}}$.

In each of these three possible ways, the matrix (W, or X, or Y) will be close to the identity matrix, and we could then follow Wolfenstein and parametrize them accordingly. In each of these cases, we will write the orthogonal matrix W, or X, or Y in the form $e^A \simeq I + A + \frac{1}{2}A^2$ with

$$A = \begin{pmatrix} 0 & \alpha & -\gamma \\ -\alpha & 0 & \beta \\ \gamma & -\beta & 0 \end{pmatrix} \tag{17}$$

We will discuss each of these possibilities in turn.

(I) We solve the 3 equations $W_{jj} = (V^T V_{\exp}^{\mathrm{corr}})_{jj}$, $j = 1, 2, 3$ (no repeated summation convention) for the 3 unknowns α, β, and γ. To the order indicated, we have 3 coupled quadratic equations and we obtain 8 solutions corresponding to various choices of signs. We then compute $\sum_{i \neq j}(W_{ij} - (V^T V_{\exp}^{\mathrm{corr}})_{ij})^2$ and determine the solution which minimizes this quantity. In this way, we find that the best fit to W is given by $\alpha = -0.00249, \beta = 0.0440$, and $\gamma = 0.0374$. The main outcome of this simple phenomenological analysis is that α is an order of magnitude smaller than β and γ, which appear to be comparable. Indeed, we see that α is close to $-3\beta^2/2$. In the spirit of Wolfenstein's paper, without any theoretical understanding whatsoever of the values of α, β, and γ, we could try, instead of a 3-parameter fit, a 2-parameter fit by setting

$$A = \begin{pmatrix} 0 & -3\nu\beta^2/2 & -\beta \\ 3\nu\beta^2/2 & 0 & \beta \\ \beta & -\beta & 0 \end{pmatrix} \tag{18}$$

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where we expect ν to be of order 1. Solving the 2 equations $W_{12} = (V^T V_{\exp}^{\mathrm{corr}})_{12}$ and $W_{23} = (V^T V_{\exp}^{\mathrm{corr}})_{23}$, we find that $\nu = 0.908$ and $\beta = 0.0439$.

(II) The analysis clearly proceeds in the same way: we solve the 3 equations $X_{jj} = (V_{\exp}^{\mathrm{corr}} V^T)_{jj}$ and obtain the best fit solution $\alpha = 0.0413$, $\beta = -0.0143$, and $\gamma = 0.0378$.

(III) In this case we solve the 3 equations $Y_{jj} = (V_{23}^T V_{\exp}^{\mathrm{corr}} V_{12}^T)_{jj}$ and obtain the best fit solution $\alpha = 0.00249$, $\beta = 0.0143$, and $\gamma = -0.0559$. In other words, we have to rotate ν_e, $\nu_x \equiv (\nu_\mu + \nu_\tau)/\sqrt{2}$ and $\nu_y \equiv (\nu_\mu - \nu_\tau)/\sqrt{2}$ by a tiny amount before mixing them with V_{12}^T.

IV. Another Ansatz for the Neutrino Mixing Matrix

Giunti [15] has proposed another interesting Ansatz for the mixing matrix

$$V_G = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{2\sqrt{2}} & \frac{\sqrt{3}}{2\sqrt{2}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{19}$$

which also factorizes nicely $V_G = V_{23} V_{G12}$ where

$$V_{G12} = \begin{pmatrix} -\frac{\sqrt{3}}{2} & \frac{1}{2} & 0 \\ \frac{1}{2} & \frac{\sqrt{3}}{2} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{20}$$

Thus, V_G and V differ by a rotation of about 5° in the $(1-2)$ plane:

$$V_G = V \begin{pmatrix} \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} & \frac{1}{2} - \frac{1}{\sqrt{6}} & 0 \\ -\frac{1}{2} + \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{2}} + \frac{1}{2\sqrt{3}} & 0 \\ 0 & 0 & 1 \end{pmatrix}. \tag{21}$$

Numerically,

$$V_G = \begin{pmatrix} -0.866 & 0.5 & 0 \\ 0.354 & 0.612 & 0.707 \\ 0.354 & 0.612 & -0.707 \end{pmatrix}, \tag{22}$$

which is to be compared to V in (16). We see that V_G makes the $\mu 1$ and $\tau 1$ entries smaller (0.354 rather than 0.408) and the $\mu 2$ and $\tau 2$ entries larger (0.612 rather than 0.577).

We could ask whether V or V_G is closer to the experimental mixing matrix $V_{\exp}^{\mathrm{corr}}$. The group invariant distance between two group elements O_1 and O_2 of $SO(3)$ is given by

$$d_{12} \equiv 3 - Tr(O_1^T O_2) \tag{23}$$

We find

$$d(V) \equiv 3 - Tr(V^T V_{\exp}^{\mathrm{corr}}) \simeq 0.00334 \tag{24}$$

and

$$d(V_G) \equiv 3 - Tr(V_G^T V_{\exp}^{\mathrm{corr}}) \simeq 0.0122, \tag{25}$$

about a factor of 4 larger than $d(V)$. If (and only if) we believe in the more or less arbitrary way in which $V_{\exp}^{\mathrm{corr}}$ is extracted from the raw data (1), then V is a somewhat better fit to the data than V_G.

V. Angular Parametrization of the Neutrino Mixing Matrix

We could of course always parametrize the neutrino mixing matrix with three Euler angles and a phase. The factorization in (7) suggests that we use the analog of the Chau-Keung [16] parametrization commonly used for the

Cabibbo-Kobayashi-Maskawa matrix in the quark sector and thus we write (where as usual $c_{23} \equiv \cos\theta_{23}$, $s_{23} \equiv \sin\theta_{23}$, and so forth)

$$V_{\text{angular}} = V_{23}V_{31}V_{12} \tag{26}$$

$$= \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_{23} & s_{23} \\ 0 & s_{23} & -c_{23} \end{pmatrix} \begin{pmatrix} c_{31} & 0 & s_{31}e^{-i\phi} \\ 0 & 1 & 0 \\ -s_{31}e^{i\phi} & 0 & c_{31} \end{pmatrix} \begin{pmatrix} -c_{12} & s_{12} & 1 \\ s_{12} & c_{12} & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{27}$$

$$= \begin{pmatrix} -c_{31}c_{12} & c_{31}s_{12} & s_{31}e^{-i\phi} \\ s_{12}c_{23} + c_{12}s_{23}s_{31}e^{i\phi} & c_{12}c_{23} + s_{12}s_{23}s_{31}e^{i\phi} & s_{23}c_{31} \\ s_{12}s_{23} - c_{12}c_{23}s_{31}e^{i\phi} & c_{12}s_{23} + s_{12}c_{23}s_{31}e^{i\phi} & -c_{23}c_{31} \end{pmatrix} \tag{28}$$

(Our parametrization differs slightly from that of Chau and Keung in that we take $\det V_{23} = \det V_{12} = -1$ in accordance with our earlier sign choice.) Evidently, for the Ansatz in (6) and in (19) $\theta_{31} = 0$ and thus $V_{e3} = s_{31}e^{-i\phi} = 0$.

If we take $V_{\text{exp}}^{\text{corr}}$ seriously, then a fit to V_{angular} gives

$$\theta_{12} = 0.617 = 35.4^o; \theta_{23} = 0.800 = 45.8^o; \theta_{31} = 0.056 = 3.21^o. \tag{29}$$

These values are extremely close by construction to the corresponding values for the Ansatz V_{HPSHZ} in (6)

$$\theta_{12} = 35.3^o; \theta_{23} = 45^o; \theta_{31} = 0^o. \tag{30}$$

This discussion makes clear how easy it is to construct more theoretical Ansatzes. Denote the three columns in the theoretical V by $\vec{v}_1, \vec{v}_2$, and $\vec{v}_3$ respectively. Pick one of the columns, call it $\vec{v}$, compare with the corresponding column in $V_{\text{exp}}^{\text{corr}}$ and fit two of the components in $\vec{v}$ to some combinations of (due to some possibly unjustifiable theoretical prejudice) small integers and their square roots. The third component of $\vec{v}$ is then fixed by orthonormality. Pick another column, call it $\vec{v}'$, and fit one of the components in $\vec{v}'$ to some combinations of small integers and their square roots. The other two components of $\vec{v}'$ are then fixed by orthonormality and by orthogonality to $\vec{v}$; they emerge as solutions of a quadratic equation. The last column is then fixed to be $\vec{v}'' = \vec{v} \times \vec{v}'$ up to a sign.

V. To Make Some Experimentalists Happy

A number of major experiments to detect CP violation in neutrino oscillation are being planned. Unfortunately, if

V_{e3} vanishes the observable signal for CP violation also vanishes, as is shown clearly in (26). The vanishing of V_{e3},

however, is the one common feature of V_{HPSHZ} and V_G. Thus, the experimentalists involved have a human tendency to strongly dislike both V_{HPSHZ} and V_G.

Suppose, in order to make some experimentalists happy, we try to construct an alternative Ansatz V_H with $(V_H)_{e3} = 1/5$, close to its experimental upper bound (1) of 0.22. Of course at each step of the construction of V_H we will have to make an arbitrary choice. We will choose what I consider to be the most reasonable. Suppose we want $(V_H)_{\mu3}$ and $(V_H)_{\tau3}$ to remain equal in magnitude. This fixes the last column of (V_H) to be $\{1, 2\sqrt{3}, -2\sqrt{3}\}/5$. Note that $(V_H)_{\mu3} = 2\sqrt{3}/5$ remains very close to $1/\sqrt{2}$, its value in both V and V_G, since its square is equal to $12/25$. Let us next suppose $(V_H)_{e2} = 1/\sqrt{3}$ as suggested by experiment. Orthogonality then fixes $(V_H)_{\mu2}$ and $(V_H)_{\tau2}$ to be $(\sqrt{141} - \sqrt{3})/12\sqrt{3}$ and $(\sqrt{141} + \sqrt{3})/12\sqrt{3}$ respectively. The first column is completely fixed to be the vector cross product of the second and third columns. We obtain what some might consider a rather "ugly" matrix

$$V_H = \begin{pmatrix} -\frac{1}{5}\sqrt{\frac{47}{3}} & \frac{1}{\sqrt{3}} & \frac{1}{5} \\ \frac{5}{12}+\frac{\sqrt{47}}{60} & \frac{\sqrt{141}-\sqrt{3}}{12\sqrt{3}} & \frac{2\sqrt{3}}{5} \\ \frac{5}{12}-\frac{\sqrt{47}}{60} & \frac{\sqrt{141}+\sqrt{3}}{12\sqrt{3}} & -\frac{2\sqrt{3}}{5} \end{pmatrix} \tag{31}$$

Theoretically, even if we were optimistic enough to imagine that we could derive V or V_G (from some discrete symmetry group [14] for example) it is hard to imagine how we could possibly derive V_H. However, V_H provides a reasonable fit to experiment by construction; numerically

$$V_H = \begin{pmatrix} -0.792 & 0.577 & 0.200 \\ 0.531 & 0.488 & 0.693 \\ 0.302 & 0.655 & -0.693 \end{pmatrix} \tag{32}$$

to be compared with (15). We note that

$$d(V_H) \equiv 3 - Tr(V_H^T V_{\exp}^{\text{corr}}) \simeq 0.0214, \tag{33}$$

about a factor of 2 larger than $d(V_G)$. This somewhat meaningless exercise merely shows that it is perfectly consistent with present data to have a large $e3$ matrix element, but it also illustrates the fact, which the reader can convince himself or herself by playing around, that it is not easy to come up with a "nice" theoretical Ansatz for the mixing matrix. Orthogonality typically generates square roots of large integers, and by going through a number of cases, I see that this usually happens unless $V_{e3} = 0$ as in V_{HPSHZ}, as may be obvious to the reader.

VI. The Value of V_{e3}

As mentioned earlier, the value of V_{e3} is of crucial interest to some experimentalists. In the last section we simply put in the value $1/5$, and in both the theoretical Ansatzes (6) and (19) V_{e3} is set to 0. We could perhaps turn things around and try to "predict" V_{e3}.

We will try to find another Ansatz, which we will call V_C, by fitting 3 numbers from the first and second columns of $V_{\exp}^{\text{corr}}$, and see what we get for V_{e3}. For the reader's convenience I reproduce $V_{\exp}^{\text{corr}}$ here:

$$V_{\exp}^{\text{corr}} = \begin{pmatrix} -0.814 & 0.578 & 0.056 \\ 0.436 & 0.545 & 0.716 \\ 0.384 & 0.607 & -0.696 \end{pmatrix}. \tag{34}$$

Let us look at column 2. As I already noted in constructing (6), $\frac{1}{\sqrt{3}} \simeq 0.577$ is very close to $(V_{\exp}^{\text{corr}})_{e2}$ and so we set $(V_C)_{e2} = \frac{1}{\sqrt{3}}$. But in contrast to the construction of (6), here we will take seriously the apparent fact that $(V_{\exp}^{\text{corr}})_{\mu 2}$ is somewhat smaller than $(V_{\exp}^{\text{corr}})_{e2}$: with a bit of fiddling we find that $\frac{2}{3}\sqrt{\frac{2}{3}} \simeq 0.544$ gives a good fit to $(V_{\exp}^{\text{corr}})_{\mu 2}$. Orthonormality then fixes $(V_{\exp}^{\text{corr}})_{\tau 2}$ to be $\frac{1}{3}\sqrt{\frac{10}{3}} \simeq 0.609$. Now that we have filled column 2 of V_C we can only fix one of the numbers in column 1: orthonormality of column 1 and orthogonality between column 1 and column 2 will fix the other two. We observe that $\frac{2}{3}\sqrt{\frac{1}{3}} \simeq 0.385$ is very close to $(V_{\exp}^{\text{corr}})_{\tau 1}$, so let's set $(V_C)_{\tau 1} = \frac{2}{3}\sqrt{\frac{1}{3}}$. We now have to solve a quadratic equation to determine $(V_C)_{e1}$ and $(V_C)_{\mu 1}$, and here comes a "problem": as in the last section, there is no reason that the solution of a generic quadratic equation would involve only small integers and the square root of small integers. In fact, already $\sqrt{10}$ has appeared. Indeed, we obtain $(V_C)_{e1} = -\frac{2}{51}(3\sqrt{26} + \sqrt{30})$ and $(V_C)_{\mu 1} = \frac{1}{153}(27\sqrt{13} - 8\sqrt{15})$. The vector $\vec{v}_3$ in column 3 is now determined to be the vector cross product of the vectors $\vec{v}_1$ and $\vec{v}_2$ in column 1 and column 2, and we obtain the Ansatz

$$V_C = \begin{pmatrix} -\frac{2}{51}(3\sqrt{26} + \sqrt{30}) & \frac{1}{\sqrt{3}} & \frac{\sqrt{2}}{51}(-12 + \sqrt{195}) \\ \frac{1}{153}(27\sqrt{13} - 8\sqrt{15}) & \frac{2}{3}\sqrt{\frac{2}{3}} & \frac{2}{153}(27 + 2\sqrt{195}) \\ \frac{2}{3}\sqrt{\frac{1}{3}} & \frac{1}{3}\sqrt{\frac{10}{3}} & -\frac{1}{3}\sqrt{\frac{13}{3}} \end{pmatrix}. \tag{35}$$

Thus, we "predict" V_{e3} to be

$$\frac{\sqrt{2}}{51}(\sqrt{195} - 12) \simeq 0.0544677, \tag{36}$$

which "happens" to be in a range encouraging to experimentalists. Numerically,

$$V_C = \begin{pmatrix} -0.815 & 0.577 & 0.055 \\ 0.434 & 0.544 & 0.718 \\ 0.385 & 0.608 & -0.694 \end{pmatrix}. \tag{37}$$

By inspection we see that (37) gives a good fit to (34); this is confirmed by the group invariant distance between V_C and $V_{\exp}^{\text{corr}}$:

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$$d(V_C) \equiv 3 - Tr(V_C^T V_{\text{exp}}^{\text{corr}}) \simeq 1.08 \times 10^{-5}. \tag{38}$$

Compare $d(V_C)$ with $d(V)$ and $d(V_G)$.

Of the three Ansatzes (I am discounting V_H) V_C emerges as the champion. However, most theorists, perhaps unjustifiably prejudiced against large integers (they might pop out of string theory?), would probably regard V_C as less elegant than the other two Ansatzes even though it gives a better fit. Thus, in a sense, this exercise reinforces my belief in, or rather, liking for the Ansatz V in (6).

Whenever experimentalists refine the raw data in (1) the reader can easily play this game. Extract some kind of mean value V_{mean} and then orthogonalize or unitarize it to an orthogonal or unitary $V_{\text{exp}}^{\text{corr}}$. The proposed Ansatz should be judged by its group theoretic distance to $V_{\text{exp}}^{\text{corr}}$. This procedure is further illustrated in the Appendix.

VII. Neutrino Mass Matrix

Neutrino oscillation experiments can only determine the absolute value of the mass squared differences $\Delta m_{ij}^2 \equiv m_i^2 - m_j^2$ rather than the three individual masses m_i. At the 99.3% confidence level Δm_{ij}^2 are determined to be $1.5 \times 10^{-3} eV^2 \leq |\Delta m_{32}^2| \leq 5.0 \times 10^{-3} eV^2$ and $2.2 \times 10^{-5} eV^2 \leq |\Delta m_{21}^2| \leq 2.0 \times 10^{-4} eV^2$, with the best fit values given by

$$|\Delta m_{32}^2| = 3 \times 10^{-3} eV^2 \tag{39}$$

and

$$|\Delta m_{21}^2| = 7 \times 10^{-5} eV^2 \tag{40}$$

We could have either the so-called normal hierarchy in which $|m_3| > |m_2| \sim |m_1|$ or the inverted hierarchy $|m_3| < |m_2| \sim |m_1|$. At present, we have no understanding of the neutrino masses just as we have no understanding of the charged lepton and quark masses.

Suppose we claim that we know the neutrino mixing matrix V. As before, call the three column vectors in the mixing matrix $\vec{v}_i$. The neutrino mass matrix is then given by

$$M = \sum_{i=1}^{3} m_i \vec{v}_i (\vec{v}_i)^T. \tag{41}$$

In other words, the 6 parameters of a real symmetric matrix have been reduced to the 3 eigenvalues $\{m_i\}$. In particular, if we believe in V_{HPSHZ} we have

$$M = \frac{m_1}{6} \begin{pmatrix} 4 & -2 & -2 \\ -2 & 1 & 1 \\ -2 & 1 & 1 \end{pmatrix} + \frac{m_2}{3} \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix} + \frac{m_3}{2} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix} \tag{42}$$

In [9], in discussing the neutrino mass matrix, we proposed a basis of 3 matrices other than those that appear in (42). First, the 3 column vectors in V are the eigenvectors of the matrix

$$M_0 = a \begin{pmatrix} 2 & 0 & 0 \\ 0 & -1 & 3 \\ 0 & 3 & -1 \end{pmatrix} \tag{43}$$

with eigenvalues $m_1 = m_2 = 2a$, and $m_3 = -4a$. (The parameter a merely sets the overall scale.) Thus, with M_0 as the mass matrix $\Delta m_{21}^2 = 0$ and this pattern reproduces the data $|\Delta m_{21}^2|/|\Delta m_{32}^2| \ll 1$ to first approximation.

Because of the degeneracy in the eigenvalue spectrum, V is not uniquely determined. To determine V, and at the same time to split the degeneracy between m_1 and m_2, we perturb M_0 to $M = M_0 + \delta M_D$, with the " democratic" form

$$\delta M_D = \varepsilon a \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{44}$$

The matrix δM_D is evidently a projection matrix that projects the first and third columns in V to zero. Thus, the eigenvalues are given by $m_1 = 2a, m_2 = 2a(1+3\varepsilon/2)$, and $m_3 = -4a$, where to the lowest order $\varepsilon = \Delta m_{21}^2/\Delta m_{32}^2$ and $a^2 = \Delta m_{32}^2/12$. Finally, to break the relation $|m_3| = 2|m_1| \simeq 2|m_2|$ we can always add to M a term proportional to the identity matrix.

VIII. Conclusion

In this note, we made the simple point that experimental data should be summarized and parametrized by its deviation from some suitable theoretical Ansatz, so that we are effectively dealing with an orthogonal or unitary matrix close to the identity, which we can then study in the same spirit as the Wolfenstein parametrization. Here for simplicity we have focussed on the orthogonal case, but it is straightforward to generalize our discussion to the unitary case by adding a phase. We also advocate using the group invariant distance between two matrices to measure the goodness of fit.

Acknowledgments

Appendix

In the Ansatz V_C discussed in section VI, I first set column 2 in V_C to

$$\vec{v}_2 \equiv \begin{pmatrix} V_{e2} \\ V_{\mu 2} \\ V_{\tau 2} \end{pmatrix} = \begin{pmatrix} \frac{1}{\sqrt{3}} \\ \frac{2}{3}\sqrt{\frac{2}{3}} \\ \frac{1}{3}\sqrt{\frac{10}{3}} \end{pmatrix}. \tag{45}$$

and $V_{\tau 1}$ to $\frac{2}{3}\sqrt{\frac{1}{3}}$. I then determined V_{e1} and $V_{\mu 1}$ in

$$\vec{v}_1 \equiv \begin{pmatrix} V_{e1} \\ V_{\mu 1} \\ V_{\tau 1} \end{pmatrix} \tag{46}$$

by solving the quadratic equation resulting from the conditions $\vec{v}_1 \cdot \vec{v}_1 = 1$ and $\vec{v}_1 \cdot \vec{v}_2 = 0$. Finally,

$$\vec{v}_3 \equiv \begin{pmatrix} V_{e3} \\ V_{\mu 3} \\ V_{\tau 3} \end{pmatrix} \tag{47}$$

was determined by $\vec{v}_3 = \vec{v}_1 \times \vec{v}_2$. "Large" integers and a non-zero value of V_{e3} emerge.

Now suppose, instead of fixing $V_{\tau 1}$ in $\vec{v}_1$, I follow the two alternate possibilities: (A) fix V_{e1} to $-\sqrt{\frac{2}{3}} \simeq -0.817$, or (B) fix $V_{\mu 1}$ to $\frac{\sqrt{3}}{4} \simeq 0.433$. Note for comparison that $(V_{\exp}^{\mathrm{corr}})_{e1} = -0.814$ and $(V_{\exp}^{\mathrm{corr}})_{\mu 1} = 0.436$. For possibility (A) and possibility (B), let me ask you three questions. (1) Does a non-zero value of V_{e3} emerge? (2) Do "large" integers appear? (3) How close is the resulting Ansatz to $V_{\exp}^{\mathrm{corr}}$?

The point is that, at least for me, it is not easy to answer these questions without doing the computation (it is of course easy to write a short program to perform the computation).

It turns out that

$$V_A = \begin{pmatrix} -\sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ \frac{2}{3}\sqrt{\frac{1}{3}} & \frac{2}{3}\sqrt{\frac{2}{3}} & \frac{\sqrt{5}}{3} \\ \frac{1}{3}\sqrt{\frac{5}{3}} & \frac{1}{3}\sqrt{\frac{10}{3}} & -\frac{2}{3} \end{pmatrix} \tag{48}$$

has a "nice" form without "large" integers and predicts $V_{e3} = 0$, with a distance from $V_{\exp}^{\mathrm{corr}}$ of

$$d(V_A) \equiv 3 - Tr(V_A^T V_{\exp}^{\mathrm{corr}}) \simeq 0.00484, \tag{49}$$

better than V_G and slightly worse than V_{HPSHZ}. Meanwhile, V_B contains huge integers and is too "ugly" to write down here, but predicts

$$V_{e3} = \frac{1}{228}(27\sqrt{10} - 2\sqrt{1338}) \simeq 0.0536 \tag{50}$$

in a range encouraging to experimentalists, and deviates from $V_{\exp}^{\mathrm{corr}}$ by only

$$d(V_B) \equiv 3 - Tr(V_B^T V_{\exp}^{\mathrm{corr}}) \simeq 1.78 \times 10^{-5}, \tag{51}$$

much better than both V_G and V_{HPSHZ} and only somewhat worse than V_C.

It appears that the absence of large integers and $V_{e3} = 0$ are correlated. If we take $V_{\exp}^{\mathrm{corr}}$ seriously and if we are not prejudiced against large integers we can get a better fit with a non-vanishing V_{e3}.

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